ADDITIVE AND MULTIPLICATIVE MAGIC CUBES

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A *magic square* (a square array containing natural numbers $1, 2, \ldots, n^2$ such that the sum of the numbers along every row and column and every diagonal is the same) has fascinated people for centuries. On Figure 1 are depicted three square tables 3×3, 4×4 a 5×5 containing different natural numbers in such a way that the product of numbers in every row, column and diagonal is the same. (In the first table the product is 6^3, in the second 7! and in the third 9!.) Such tables are called *multiplicative magic squares.*

12	1	18
9	6	4
2	36	3

1	24	14	15
21	10	4	6
20	7	18	2
12	3	5	28

1	15	42	16	36
14	32	9	3	30
27	6	10	28	8
20	7	24	54	2
48	18	4	5	21

FIGURE 1 - MULTIPLICATIVE MAGIC SQUARES

An additive magic cube is a natural generalization of a magic square. In 1686, *Adamas Kochansky* extended magic squares to three dimensions. The first additive magic cube probably appeared in a letter of *Pierre de Fermat* from 1640. There is a lot of information and many interesting results about magic squares and cubes in the references and web-pages.

An *additive magic cube* of order n is a cubical array

$$\mathbf{M}_n = |\mathbf{m}_n(i, j, k); \quad 1 \leq i, j, k \leq n|$$

containing natural numbers $1, 2, 3, \ldots, n^3$ such that the sum of the numbers along every row and diagonal is the same, i.e. $\frac{n(n^3+1)}{2}$. By a row of a magic cube we mean an n-tuple of elements having the same coordinates on two places. Every additive magic cube of order n has exactly $3n^2$ rows and 4 diagonals.

FIGURE 2 - ADDITIVE MAGIC CUBE $\mathbf{M}_3$

A *multiplicative magic cube* of order n is a cubical array

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k); \quad 1 \le i,j,k \le n|,$$

containing n^3 mutually different natural numbers such that the product of the numbers along each row and every of its four diagonals is the same. We call this product *magic constant* and denote $\sigma(\mathbf{Q}_n)$.

FIGURE 3 - MULTIPLICATIVE MAGIC CUBE $\mathbf{Q}_3$

In [7] it is proved that an additive magic cube $\mathbf{M}_n$ of order n exists for every $n \neq 2$. Constructions consider three cases (n is an odd integer, if n is an even integer then we distinguish whether if n is or is not divisible by four.)

If we know a construction of $\mathbf{M}_n = |\mathbf{m}_n(i,j,k)|$ then we can easily make a multiplicative magic cube

$$\mathbf{Q}_n = |\mathbf{q}_n(i,j,k) = 2^{\mathbf{m}_n(i,j,k)-1}; 1 \le i,j,k \le n|$$

with the magic constant $\sigma(\mathbf{Q}_n) = 2^{\frac{n(n^3-1)}{2}}$. This constant is very big and so the following question can be asked: What is the smaller magic constant of $\mathbf{Q}_n$?

By the end of the 19-th century (see [2,p.351]) mathematicians began to consider also 4-dimensional magic cubes. But only in 2001 the following result was published:

Theorem. *An additive magic d-dimensional cube of order n exists if and only if $d > 1$ and $n \neq 2$ or $d = 1$.*

In Figure 4 are depicted the nine layers of a magic 4-dimensional cube of order 3. The element $\mathbf{m}(1,1,1,1) = 46$ is in four rows containing the triplets of numbers $\{46,8,69\}$, $\{46,62,15\}$, $\{46,17,60\}$ and $\{46,59,18\}$. In the eight diagonals there are the triplets $\{46,41,36\}$, $\{69,41,13\}$, $\{15,41,67\}$, $\{35,41,47\}$, $\{60,41,22\}$, $\{26,41,56\}$, $\{53,41,29\}$ and $\{64,41,18\}$. (Note. This picture is a magic square of order 9 with some special proprieties.) This cube was constructed using the following formula

$$\begin{aligned}\mathbf{m}(i_1,i_2,i_3,i_4) = {} & [(i_1 - i_2 + i_3 - i_4 + \tfrac{n+1}{2} - 1) \pmod n]n^3 \\ & + [(i_1 - i_2 + i_3 + i_4 - \tfrac{n+1}{2} - 1) \pmod n]n^2 \\ & + [(i_1 - i_2 - i_3 - i_4 + 3\tfrac{n+1}{2} - 1) \pmod n]n \\ & + [(i_1 + i_2 + i_3 + i_4 - 3\tfrac{n+1}{2} - 1) \pmod n] + 1.\end{aligned}$$

46	8	69	17	78	28	60	37	26
62	42	19	51	1	71	10	80	33
15	73	35	55	44	24	53	6	64
59	39	25	48	7	68	16	77	30
12	79	32	61	41	21	50	3	70
52	5	66	14	75	34	57	43	23
18	76	29	58	38	27	47	9	67
49	2	72	11	81	31	63	40	20
56	45	22	54	4	65	13	74	36

FIGURE 4 - MAGIC 4-DIMENSIONAL CUBE OF ORDER 3

Similarly we can consider the existence of multiplicative magic d-dimensional cubes for any natural d.

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